

Soliton molecules in trapped vector Nonlinear Schrödinger systems

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We study a new class of vector solitons in trapped Nonlinear Schrödinger systems modelling the dynamics of coupled light beams in GRIN Kerr media and atomic mixtures in Bose-Einstein condensates. These solitons exist for different spatial dimensions, their existence is studied by means of a systematic mathematical technique and the analysis is made for inhomogeneous media.

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Since the beginning of its history, physics has studied simple objects and the way they arrange to form more complex structures. Some remarkable successes included the atomic theory of matter, the structure of nucleus in terms of protons and neutrons and the substructure of nucleons in terms of quarks to cite a few examples.

Elementary *robust* objects made of light have been known since the 70's. In fact, *spatial optical solitons*—self-trapped states of light with particle-like properties—have attracted a considerable attention during last years as possible building blocks of all-optical switching devices where light is used to guide and manipulate light itself [1, 2]. Another field where robust solitonic structures have been recently found is that of Bose-Einstein condensation where the dilute quantum gas supports robust structures such as one-dimensional dark solitons [3], bright solitons [4] or vortices [5].

In nonlinear optics, the robust nature of spatial optical solitons [2], allows to draw an analogy with atomic physics treating spatial solitons as “atoms of light”. Furthermore, when several light beams are combined to produce a *vector soliton*, this process can be viewed as the formation of composite states or “molecules of light”.

Several structures of this type have been studied previously: dipole and multipole vector solitons [6, 7, 8, 9], self-trapped necklace ring beams [10], rotating propeller solitons [11] and rotating optical soliton clusters [12].

In the field of Bose-Einstein condensation (BEC) vortex solitons arrange to generate lattice structures in rotating [13] and non-rotating scalar systems [14]. In multicomponent condensates simple stationary solutions involving some type of dynamical equilibria between the constituent solitons have been described [15, 16, 17].

The purpose of this letter is to describe and analyze in detail a way to build multidimensional “soliton molecules” in vector systems with applications to nonlinear optics and to Bose-Einstein condensed gases. As new features these solitons will be analyzed using a systematic mathematical technique, the analysis will be made for inhomogeneous media and the soliton molecules exist for different dimensions $D = 2, 3$ thus providing the first soliton molecules shown to exist for $D = 3$.

The model.— We will consider a system of N complex

fields $u_1(t, \mathbf{r}), u_2(t, \mathbf{r}), \dots, u_N(t, \mathbf{r})$ ruled by the equations

$$i\partial_t u_j(t, \mathbf{r}) = \left[-\frac{1}{2}\Delta + V(\mathbf{r}) + U_j(t, \mathbf{r}) \right] u_j(t, \mathbf{r}), \quad (1)$$

for $j = 1, \dots, N$. The coupling term is given by $U_j(t, \mathbf{r}) = \sum_k g_{jk} |u_k(t, \mathbf{r})|^2$ with $g_{jk} \in \mathbb{R}$. Eqs. (1) are a set of Nonlinear Schrödinger equations (NSE) which in BEC problems describe multicomponent systems, u_j being the wavefunctions for each of the atomic species involved [15, 16, 17]. In optics these equations describe the incoherent interaction between the slowly varying envelopes of the electric field in paraxial beams in Kerr media. We choose $V(\mathbf{r}) = \mathbf{r}^2/2$ which corresponds to an isotropic magnetic trapping in BEC and to a GRIN fiber in the optical case. *Single component case: Soliton atoms.*— Let us first consider the scalar case ($N = 1$). Solitons or stationary solutions of Eq. (1) in the scalar case have the form $u(t, \mathbf{r}) = \phi_\mu(\mathbf{r})e^{i\mu t}$ and satisfy

$$\mu\phi_\mu = -\frac{1}{2}\Delta\phi_\mu + \frac{1}{2}\mathbf{r}^2\phi_\mu + g|\phi_\mu|^2\phi_\mu \quad (2)$$

For a fixed norm of the solution $\|\phi\|_{L^2} := \int |\phi|^2 dV$, any solution to Eq. (2) will be a valid soliton atom for our purposes. The simplest case corresponds to a nodeless ground state solution which for strong interaction is close to the Thomas-Fermi solution (thus a quasi-compacton type of solution) [17] and in the small interaction case (as it happens in nonlinear optics) it is close to a Gaussian function. Many other complex stationary solutions to Eq. (2) are possible such as vortices. These objects will play the role of “atoms” in what follows.

The relevant property to be used here [18], is that any function of the form

$$u(\mathbf{r}, t) = \phi_\mu(\mathbf{r} - \mathbf{R}(t)) e^{i[\mu t + \theta(\mathbf{r}, t)]}, \quad (3)$$

is a solution of the scalar time-dependent NLSE provided $\mathbf{R}(t)$ satisfies $\frac{d^2}{dt^2}\mathbf{R} + \mathbf{R} = 0$ and $\theta(\mathbf{r}, t) = (\mathbf{r}, \frac{d}{dt}\mathbf{R}) + f(t)$, where $f(t) = \int_0^t \left[\left(\frac{d}{dt}\mathbf{R} \right)^2 - \mathbf{R}^2 \right] dt$. This means that exact scalar time-dependent solutions whose center evolves according to harmonic oscillator type equations and *preserve the shape of the stationary solution* during evolution can be built. Remarkably, this property is not exclusive of the Kerr nonlinearity and it is valid for any

type of nonlinear term for which localized solutions exist. However, if $N \geq 2$, the property described previously does not hold except for very trivial situations. Because of the cross-interaction. In a general situation, the pulses collide and lose their individuality. Here we want to give some ideas on how to overcome this problem and build stationary nontrivial vector solitons of Eqs. (1).

Formalism for the multicomponent case.- Let us define the modulus, n_j , and phase, ϕ_j , of each wavepacket through $u_j = \sqrt{n_j} \exp(i\phi_j)$. Let us define also the center of mass of species j , $\mathbf{R}_j(t) = \int dV n_j \mathbf{r}$ and their total momenta $\mathbf{P}_j(t) = \int dV n_j \nabla \phi_j$, whose evolution laws are

$$\frac{d}{dt} \mathbf{R}_j = \mathbf{P}_j, \quad (4)$$

$$\frac{d}{dt} \mathbf{P}_j = -\mathbf{R}_j + \frac{1}{2} \sum_k g_{jk} \mathbf{F}_{jk} \quad (5)$$

The first term in the r.h.s of (5) corresponds to the external potential, while the nonlinear force is given by

$$\mathbf{F}_{jk} = \int dV (n_k \nabla n_j - n_j \nabla n_k) \quad (6)$$

Generally \mathbf{F}_{jk} may be a rather complex function, but if the n_j for $j = 1, \dots, N$, have small enough overlapping, it is possible to argue that these forces would be central. To prove this affirmation let us first notice that for the scalar case and far from the center of the wavepacket, the self-interaction is small and u can be described by the linear theory to be $n(\mathbf{r}) \propto e^{-r^2} [r^{2m} + \mathcal{O}(r^{2m-2})]$ for $r \rightarrow \infty$. Similar considerations apply to the multicomponent case. If the wavepackets are separated, then in the region where the u_j overlap (i.e., the region which determines the value of the integral in (6)) the nonlinear terms will be small and

$$\nabla n_j = -2(\mathbf{r} - \mathbf{R}_j) + \mathcal{O}\left(\frac{1}{|\mathbf{r} - \mathbf{R}_j|}\right), \quad (7)$$

from which it follows that

$$\mathbf{F}_{jk} = 2(\mathbf{R}_j - \mathbf{R}_k) \int dV n_j n_k + \mathcal{O}\left(\frac{1}{|\mathbf{R}_k - \mathbf{R}_k|}\right), \quad (8)$$

i.e., if the wavepackets are separated, in the leading order the *inter-mode force is central*.

As to the factor $\int dV n_j n_k$, let us evaluate it, for illustrative purposes, for the case when $u_{j,k}$ are solutions of linear problem ($g_{jk} = 0$) of the type $n_j = N_j |\mathbf{r} - \mathbf{R}_j|^{2m_j} e^{-(\mathbf{r} - \mathbf{R}_j)^2}$, where N_j are normalization constants (these functions include the fundamental mode $m_j = 0$ and vortex states)

$$\int dV n_j n_k = \frac{\pi}{2} N_j N_k e^{-\frac{1}{2}|\mathbf{R}_j - \mathbf{R}_k|^2} K_{m_j, m_k}(|\mathbf{R}_j - \mathbf{R}_k|), \quad (9)$$

where K_{m_j, m_k} are polynomial factors, the lowest order ones for $D = 2$ being $K_{0,0}(Q) = 1$, $K_{0,1}(Q) = Q^2/4$

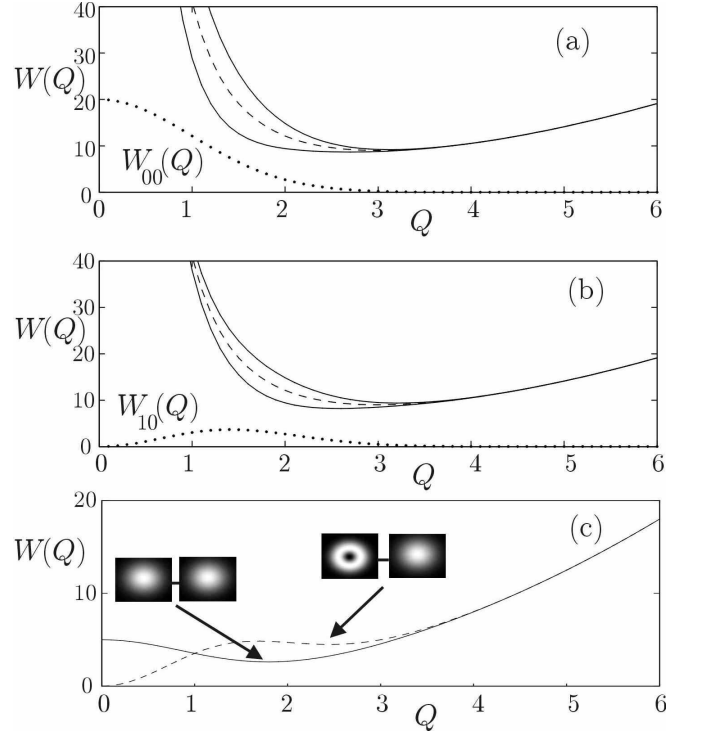


FIG. 1: Potential $W(Q)$ for $g_{jk} = 0$ (dashed line) and nonlinear potential $W_f(Q)$ (dotted line) for: (a) Interaction of two gaussian modes, (b) Interaction of a gaussian with a vortex mode. Solid lines represent the total effective potential for $\bar{g}/2\pi = 20$ (upper solid line) or $\bar{g}/2\pi = -20$ (lower solid line) (c) A situation with $L = 0$. Solid line: total potential for two gaussian modes with $\bar{g}/2\pi = 5$. Dashed line: potential for gaussian plus vortex modes with $\bar{g}/2\pi = 20$. The arrows mark the minima corresponding to non-rotating vector molecules

and $K_{1,1}(Q) = [2 - Q^2 + \frac{1}{4}Q^4]/4$. Thus, in the case of small nonlinearities, we can estimate the force between the wavepackets by using Eq. (9). For our purposes the specific form of the interaction is less crucial than the fact that the forces \mathbf{F}_{jk} are central.

Two-component case.- For the two-component symmetric ($g_{12} = g_{21} = \bar{g}$) case Eqs. (5) read

$$\frac{d}{dt} \mathbf{P}_1 = -\mathbf{R}_1 + \frac{1}{2} \bar{g} \mathbf{F}, \quad (10a)$$

$$\frac{d}{dt} \mathbf{P}_2 = -\mathbf{R}_2 - \frac{1}{2} \bar{g} \mathbf{F}, \quad (10b)$$

where $\mathbf{F} = \int dV (n_2 \nabla n_1 - n_1 \nabla n_2)$. The modified 'total' center of mass $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2$ and momentum $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ do not 'feel' the force \mathbf{F} since they satisfy $\frac{d}{dt} \mathbf{R} = \mathbf{P}$ and $\frac{d}{dt} \mathbf{P} = -\mathbf{R}$, i.e. they oscillate harmonically.

The most interesting quantity is $\mathbf{Q} = \mathbf{R}_1 - \mathbf{R}_2$ (for the general multicomponent case we define $\mathbf{Q}_{ij}(t) = \mathbf{R}_i(t) - \mathbf{R}_j(t)$), which gives the separation between the centers of mass of the two components and evolves according to

$$\frac{d^2 \mathbf{Q}}{dt^2} + \mathbf{Q} = \bar{g} \mathbf{F} \quad (11)$$

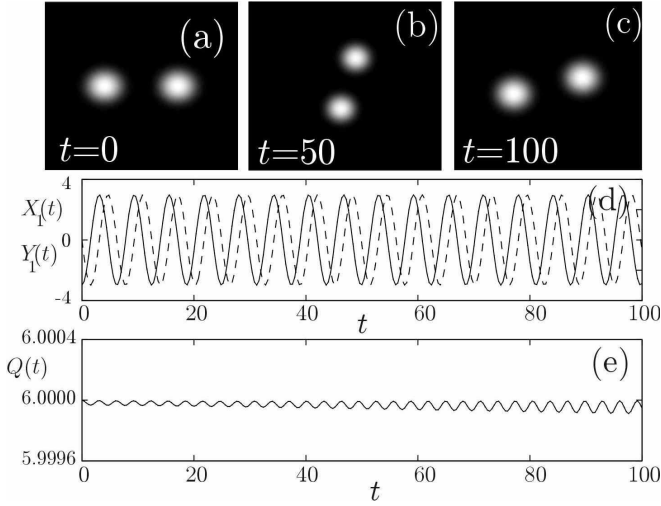


FIG. 2: Evolution of initial data $u_1(\mathbf{r}, t) = \phi_0(\mathbf{r} - \mathbf{R}_1(0))e^{3iy}$, $u_2(\mathbf{r}, t) = \phi_0(\mathbf{r} + \mathbf{R}_1(0))e^{-3iy}$ with $g_{11} = g_{12} = g_{22} = 10$, $\int n_j dV = 1$, $\mathbf{R}_1(0) = (3, 0)$ and ϕ_0 is the scalar ground state. (a-c) Density plots of $n_1(\mathbf{r}, t) + n_2(\mathbf{r}, t)$ on the spatial region $[-8, 8] \times [-8, 8]$. (d) Evolution of $\mathbf{R}_1(t)$: $X_1(t)$ (solid) and $Y_1(t)$ (dashed). (e) Evolution of $Q(t)$.

As discussed above, if \mathbf{Q} is sufficiently large, the force $\mathbf{F} \propto \mathbf{Q}$ and it can be presented in potential form $\mathbf{F} = -\frac{\partial}{\partial \mathbf{Q}} W_f(Q) = -W'_f(Q)\mathbf{Q}/Q$ and then Eq. (11) reads

$$\frac{d^2 \mathbf{Q}}{dt^2} + W'_{\text{tot}}(Q) \frac{\mathbf{Q}}{Q} = 0 \quad (12)$$

where $W_{\text{tot}} = \frac{1}{2}Q^2 + \bar{g}W_f(Q)$. It is interesting to note that in the approximation of independent wavepackets the potential W_f is given by $W_f(Q) = \int dV n_1 n_2$.

It is evident that Eq. (12) has two constants of motion: energy $E = \frac{1}{2}(\dot{\mathbf{Q}}, \dot{\mathbf{Q}}) + W_{\text{tot}}(Q)$ and angular momentum, $\mathbf{L} = \mathbf{Q} \times \dot{\mathbf{Q}}$. Since $\dot{\mathbf{Q}}^2 = \dot{Q}^2 + L^2/Q^2$, we can write $E = \frac{1}{2}\dot{Q}^2 + W_{\text{eff}}(Q)$, where $W_{\text{eff}}(Q) = \frac{1}{2}Q^2 + \frac{L^2}{2Q^2} + \bar{g}W_f(Q)$. There will exist an equilibrium distance Q_0 , for which the effective potential W_{eff} is minimized. The reason is that the centrifugal contribution is divergent for $Q \simeq 0$ while the trap potential is unbounded for $Q \rightarrow \infty$ and the effective nonlinear interaction should decay for large Q and have a maximum finite amplitude [Fig. 1].

In particular, if the nonlinear interaction is small then $Q_0^{\text{lin}} \simeq L^{1/2}$ thus leading to a stationary rotating solution of the vector system (1) provided the distances between the components are kept large enough. The larger the nonlinear terms the larger will the deviation of Q_0 from Q_0^{lin} be ($Q_{\text{eq}} > Q_0$ for $g_{jk} > 0$ and $Q_{\text{eq}} < Q_0$ for $g_{jk} < 0$) [Fig 1(a,b)]. When $L = 0$, the combination of the trap force and the nonlinear term may also have minima [Fig. 1(c)] corresponding to non-rotating soliton molecules. However, due to the fact that the equilibrium distance is comparable to the wavepacket's widths, our result must be taken only as an indication that such soli-

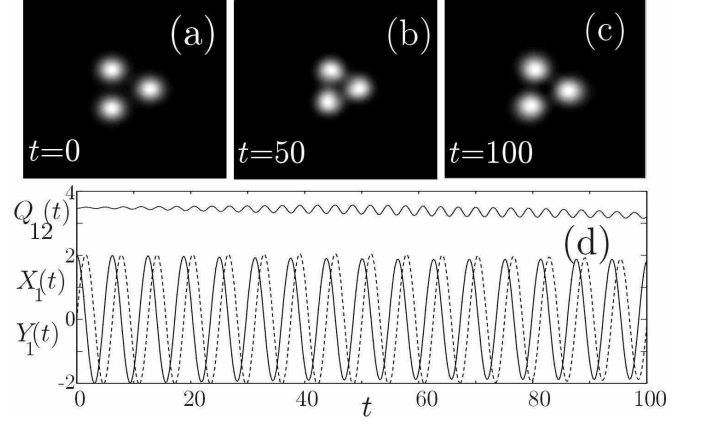


FIG. 3: Evolution of a three-component system with $u_j(\mathbf{r}, 0) = (1/\sqrt{\pi}) e^{-((x-x_j)^2 + (y-y_j)^2)/2} e^{i(v_{xj}x + v_{yj}y)}$, $(x_j, y_j) = d_0(\cos(2\pi j/3), \sin(2\pi j/3))$, $(v_{xj}, v_{yj}) = d_0(\sin(2\pi j/3), -\cos(2\pi j/3))$ for $j = 0, 1, 2$. Parameter values: $g_{ij} = 3$, $d_0 = 2$, $\int n_j dV = 1$. (a-c) Density plots of $\sum_{j=1}^3 n_j(\mathbf{r}, t)$ (d) Evolution of \mathbf{R}_1 : $X_1(t)$ (solid) and $Y_1(t)$ (dashed). (e) Evolution of $Q(t)$.

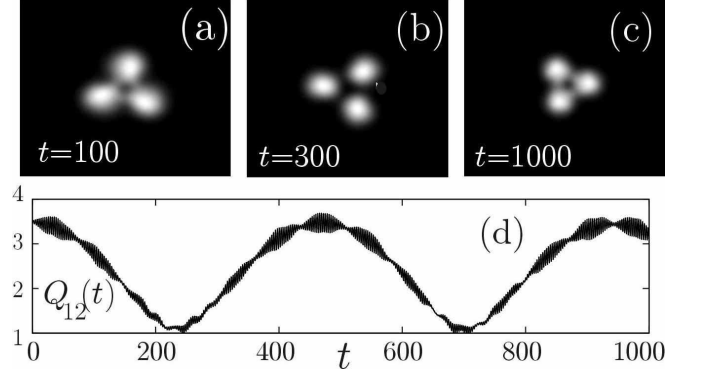


FIG. 4: Same as Fig. 3 but with $g_{ij} = 5$. (a-c) Density plots of $\sum_{j=1}^3 n_j(\mathbf{r}, t)$. (d) Long-time evolution of $Q_{12}(t)$.

ton molecules exist and not as a proof that they will be well approximated by combinations of scalar solitons.

Examples of soliton molecules.- Let us now present several examples of the soliton molecules discussed previously. First we have studied the case of a pair of weakly interacting soliton atoms. The results, obtained with a symplectic second order in time split-step integrator are summarized in Fig. 2 where it is seen how the small interaction induces only small harmonic oscillations of $Q(t)$ without appreciable distortion of the wavepackets.

In Fig. 3 three gaussian solitons interact more strongly due to the larger number of components and the smaller distance between the beams. In this case the oscillations of the distances between components, Q_{ij} , remain small [Fig. 3(d)] although some oscillation of the positions of the beams is appreciable [Fig. 3(a-c)]. Although this particular configuration is stable, the present example is a three-body problem for which many behaviors are pos-

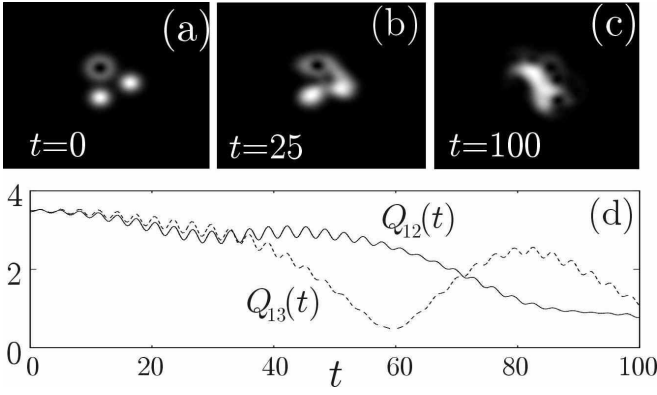


FIG. 5: Same as Fig. 4 but with a gaussian soliton replaced by a vortex soliton. (a-c) Density plots of $\sum_{j=1}^3 n_j(\mathbf{r}, t)$. (d) Evolution of the inter-mode distances $Q_{12}(t)$ (solid), and $Q_{13}(t)$ (dashed).

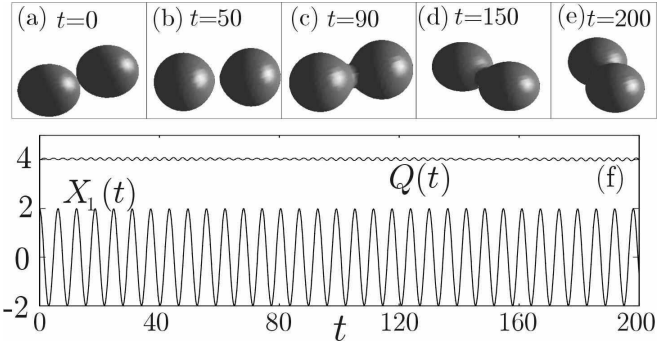


FIG. 6: Weak interaction of two gaussian solitons $u_1(\mathbf{r}, 0) = (1/\pi^{3/4})e^{-((x-2)^2+y^2+z^2)/2}e^{-2iy}$, $u_2(\mathbf{r}, 0) = (1/\pi^{3/4})e^{-((x+2)^2+y^2+z^2)/2}e^{2iy}$ with $g_{jk} = 10$. (a-e) Isosurface plots of $n_1(\mathbf{r}, t) + n_2(\mathbf{r}, t)$ for $n_1 + n_2 = 0.01$. (f) Evolution of the inter-mode distance $Q(t)$ (upper solid line), and $X_1(t)$ (lower solid line).

sible: stable solutions, resonances, chaos, etc. In fact, if the values of the nonlinear coefficient are increased the beams deform and the inter-mode distances $Q(t)$ suffer strong oscillations [Fig. 4(d)] although the structure remains stable after long periods of time containing about one hundred revolutions of the soliton around the center. Finally, in Fig. 5, it is shown that if one of the gaussian solitons is replaced by a vortex soliton the asymmetry of the interaction and the longer interaction range of the vortex soliton (for which $n \sim r^2 e^{-r^2}$) make this configuration unstable and the initial configuration is destroyed after a few rounds.

We have also analyzed several soliton molecules in three spatial dimensions. In Fig. 6 we summarize the results for the situation of a stable configuration made of two weakly interacting gaussian solitons. The evolution

for long times shows that the inter-component distance $Q(t)$ suffers only small oscillations [Fig. 6(f)] which manifest on the plots where more interaction is apparent [Fig. 6(c,d)].

Conclusions.- In this paper we have presented several soliton molecules built up from scalar solitons of the trapped NSE. The method presented here allows to generate many different multidimensional soliton molecules.

Between the possible applications of the vector solitons described here, in the field of optics they could be used as a way to transmit different laser beams in a single optical fibers. In any case the vector solitons presented here represent another step into the comprehension of complex objects sustained by nonlinear forces with promising applications in different fields.

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